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Numerical Experimentation and Comparison of Fuzzy Integrals

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Abstract

Sugeno and Choquet integrals have been widely studied in the literature from a theoretical viewpoint. However, the behavior of these functionals is known in a general way, but not in practical applications and in particular cases. This paper presents the results of a numerical comparison that attempts to be a basis for a better comprehension and usefulness of both integrals.
Key Words: Choquet integral, Sugeno integral, fuzzy measure

1 Introduction

Sugeno's fuzzy integral and Choquet's fuzzy integral, as functionals defined in order to evaluate a bounded function over a fuzzy measure, have been widely studied in the literature. Since its definition in 1974, Sugeno's fuzzy integral (also called Fuzzy Expected Value -FEV- when defined over probabilities) has been studied in some different contexts [7, 9], generalized [6, 11, 12] or characterized [3, 5, 10, 13] by many authors. Also, starting with Choquet's original definition by 1953 [4], monotone expectation -ME- has been studied into the context of fuzzy measures [1, 8] as a generalization of classical mathematical expectation over probability measures. Studies comparing and relating both functionals [2, 3] are also available (several works can be found in the cited authors' bibliography).

However, despite of the number of theoretical works, the behavior of these functionals is just viewed from a general perspective, interpreting Sugeno integral as a "weighted median" and Choquet integral as a "misshaped average"; in the practice, the comparison between both functionals is still restricted to the so called Sugeno's bound for probabilities, generalized for any fuzzy measure by Bolaños,

de Campos and González [2]. In this paper we carry out a numerical comparison attempting to be a reference for the use of both integrals and for the developement of future applications.

In section 2 some definitions and known results are introduced. In section 3 the experimentation carried out in this work is described. Experimental results are presented in section 4, and the paper ends with conclusions in section 5.

2 Definitions and known results

Let $\mathcal{P}(X)$ denote the set of all subsets of a set X. Over a finite set $X = \{x_1, x_2, \ldots, x_n\}$, a *fuzzy measure* is defined as a function $g : \mathcal{P}(X) \longrightarrow [0,1]$ verifying:

1. $g(\phi) = 0$
2. $g(X) = 1$
3. $A \subset B \Rightarrow g(A) \leq g(B) \quad A, B \in \mathcal{P}(X)$

Let be the measure space $(X, \mathcal{P}(X), g)$. Sugeno integral of a function $h : X \longrightarrow [0,1]$ with respect to a fuzzy measure g is defined as:

$$S_g(h) = \oint h \circ g = \bigvee_{i=1}^{n} [a_i \wedge g(H_{a_i})] \tag{1}$$

where H_a is the a-cut for h $(H_a = \{x \in X | h(x) \geq a\})$ and $a_i = h(x_i)$ for all $x_i \in X$.

Under these conditions, Choquet integral of a function h with respect to a fuzzy measure g is defined as:

$$C_g(h) = \int h \circ g = \int g(H_a) da = \sum_{i=1}^{n} (a_i - a_{i-1}) g(H_{a_i}) \tag{2}$$

with $a_0 = 0$.

The relation between both functionals is given by the following expression [2]:

$$|S_g(h) - C_g(h)| \leq \frac{1}{4} \tag{3}$$

for any fuzzy measure g and any function h.

3 Experimentation

In order to compare the value of both integrals for distinct fuzzy measures, the following process is drawn: functions h have been defined over a finite set with 10 elements, taking values randomly over the interval $[0,1]$. Then, the results of $C_g(h)$ and $S_g(h)$ for each fuzzy measure g have been statistically analyzed. In detail, the experimentation procedure is as follows:

Figure 1: $S_{g_1}(h)$ versus $C_{g_1}(h)$.

Figure 2: $S_{g_2}(h)$ versus $C_{g_2}(h)$.

- Consider a set X with 10 elements, $X = \{x_1, \ldots, x_{10}\}$
- Fix a fuzzy measure g.
- Generate randomly a sample consisting of 1000 functions $h : X \longrightarrow [0, 1]$ by using a mixed congruential pseudo random numbers generator.
- For each function h, compute the values for $S_g(h)$ and $C_g(h)$.
- A comparative test and a correlation analysis are carried out among the 1000 pairs of values obtained from $S_g(h)$ and $C_g(h)$.

This process has been repeated for different fuzzy measures:

1. One uniform probability measure:

$$g_1(x_i) = \frac{1}{10} \text{ for each } x_i \in X$$

Figure 3: $S_{g_3}(h)$ versus $C_{g_3}(h)$.

Figure 4: $S_{g_4}(h)$ versus $C_{g_4}(h)$.

2. One non uniform probability measure g_2 (with two values 0.25, three values 0.10 and five values 0.04).

3. One expansive fuzzy measure g_3 next to a probability (the measures over the sets increase in a 10 per cent with respect to the measure g_1). That is to say:

   Let g_1 be the measure for the first experiment. Given a function h, for any a-cut from h, $H_a = \{x_{a_1}, \ldots, x_{a_m}\}$, the measure $g_3(H_a)$ is calculated as follows:

$$t = \sum_{i=1}^{m} g_1(x_{a_i}) + \left(\sum_{i=1}^{m} g_1(x_{a_i})\right) \times \frac{1}{10} = 1.1 \times \left(\sum_{i=1}^{m} g_1(x_{a_i})\right)$$

$$g_3(H_a) = \begin{cases} t & if \quad t \leq 1 \\ 1 & if \quad t > 1 \end{cases} \tag{4}$$

Figure 5: $S_{g_5}(h)$ versus $C_{g_5}(h)$.

Figure 6: $S_{g_6}(h)$ versus $C_{g_6}(h)$.

Note that $g_3(\phi) = 0$ and $g_3(X) = 1$.

4. One expansive fuzzy measure g_4 far from a probability (the measures over the sets increase in a 50 per cent with respect to the measure g_1). That is:

$$t = \sum_{i=1}^{m} g_1(x_{a_i}) + \left(\sum_{i=1}^{m} g_1(x_{a_i})\right) \times \frac{5}{10} = 1.5 \times \left(\sum_{i=1}^{m} g_1(x_{a_i})\right)$$

$$g_4(H_a) = \begin{cases} t & if \quad t \leq 1 \\ 1 & if \quad t > 1 \end{cases} \tag{5}$$

It is easy to prove that $g_4(\phi) = 0$ y $g_4(X) = 1$.

5. One restrictive fuzzy measure g_5 next to a probability (the measures over the sets decrease in a 10 with respect to the measure g_1). That is:

$$t = \sum_{i=1}^{m} g_1(x_{a_i}) - \left(\sum_{i=1}^{m} g_1(x_{a_i})\right) \times \frac{1}{10} = 0.9 \times \left(\sum_{i=1}^{m} g_1(x_{a_i})\right)$$

$$g_5(H_a) = \begin{cases} 0 & if \quad H_a = \phi \\ 1 & if \quad H_a = X \\ t & in \quad other \quad cases \end{cases} \tag{6}$$

6. One restrictive fuzzy measure g_6 far from a probability (the measures over the sets decrease in a 50 per cent with respect to g_1). That is:

$$t = \sum_{i=1}^{m} g_1(x_{a_i}) - \left(\sum_{i=1}^{m} g_1(x_{a_i})\right) \times \frac{5}{10} = 0.5 \times \left(\sum_{i=1}^{m} g_1(x_{a_i})\right)$$

$$g_6(H_a) = \begin{cases} 0 & if \quad H_a = \phi \\ 1 & if \quad H_a = X \\ t & in \quad other \quad cases \end{cases} \tag{7}$$

Figure 7: $S_{g_7}(h)$ versus $C_{g_7}(h)$.

Moreover, four Sugeno's λ-measures have been considered (g_7, g_8, g_9, g_{10}) given by the following generic expression:

Given $A, B \subset X$, $A \cap B = \phi$ and $\lambda > -1$,

$$g_\lambda(A \cup B) = g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A) g_\lambda(B) \tag{8}$$

For the experiment, an upper bound equal to 1 has been considered for all g_λ-measures, the considered values of λ are -0.9 for g_7, -0.5 for g_8, 2 for g_9 and 5 for g_{10} and measures assigned to the unitary subsets of X are:

$$g_j(x_i) = \frac{1}{10} \text{ for all } x_i \in X, \quad j = 7, 8, 9, 10 \tag{9}$$

Figure 8: $S_{g_8}(h)$ versus $C_{g_8}(h)$.

The results of the experiments are shown in tables 1-5. Figures 1-10 give a graphical representation of the values of Sugeno and Choquet integrals and the fitted regression lines for each measure.

Figure 9: $S_{g_9}(h)$ versus $C_{g_9}(h)$.

4 Results

For g_1 (table 1) one can observe both integrals providing very similar averages (no significative differences). Choquet integral is more variable.

About measure g_2 (table 1), as in the previous case, both integrals show very similar averages, with more variability for Choquet's one.

For measure g_3 (table 2), experimental results show an average significatively higher for Choquet integral ($p < 0.001$) than for Sugeno's one. Variability stands

Figure 10: $S_{g_{10}}(h)$ versus $C_{g_{10}}(h)$.

	Results for g_1		Results for g_2	
	Mean	Std. Deviation	Mean	Std. Deviation
S(h)	0.500099	0.087413	0.499102	0.113516
C(h)	0.500466	0.109700	0.497244	0.135915
C(h)-S(h)	0.000367	0.048079	-0.001858	0.057919
Correlation C(h), S(h)		0.905369	0.907545	
Regression	b=0.721433	a=0.139047	b=0.757980	a=0.122201

Table 1: Results for g_1 and g_2.

slightly higher for Choquet integral.

Results for measure g_4 (table 2) show how the average of Choquet integrals is clearly higher ($p < 0.001$). Again, the variability is slightly higher for Choquet integral.

In the case of measure g_5 (table 3) the average for Sugeno integral is significatively higher ($p < 0.001$), but the variability is again slightly higher for Choquet integral.

The average obtained from Choquet integral with g_6 (table 3) is widely surpassed by Sugeno integral average ($p < 0.001$), and the variability is again slightly higher for Choquet integral.

With respect to the group of g_λ-measures (tables 4,5), one can say that Sugeno integral is significatively higher for restrictive measures (g_7, g_8), and significatively lower for the expansives ones (g_9, g_{10}); in all cases, $p < 0.001$. It is important to remark that for this class of measures, the variability of Sugeno integral is a little higher and that correlation coefficients are clearly lower, due to the greater concentration of the results.

Analyzing the regression lines fitted for each case, smaller sensibility of Sugeno integral can be observed, showing greater proximity to the theoretical mean value

	Results for g_3		Results for g_4	
	Mean	Std. Deviation	Mean	Std. Deviation
S(h)	0.523650	0.091162	0.593966	0.101550
C(h)	0.540660	0.115091	0.646925	0.120577
C(h)-S(h)	0.017010	0.049599	0.052959	0.050641
Correlation C(h), S(h)		0.910052	0.910064	
Regression	b=0.720841	a=0.133920	b=0.766457	a=0.098126

Table 2: Results for g_3 and g_4.

	Results for g_5		Results for g_6	
	Mean	Std. Deviation	Mean	Std. Deviation
S(h)	0.475613	0.084324	0.333643	0.055272
C(h)	0.460272	0.104752	0.299494	0.090752
C(h)-S(h)	-0.015342	0.047521	-0.034149	0.054687
Correlation C(h), S(h)		0.895794	0.827370	
Regression	b=0.721104	a=0.143710	b=0.503905	a=0.182726

Table 3: Results for g_5 and g_6.

	Results for g_7		Results for g_8	
	Mean	Std. Deviation	Mean	Std. Deviation
S(h)	0.450682	0.100068	0.474461	0.102634
C(h)	0.417336	0.077592	0.451082	0.081674
C(h)-S(h)	-0.033346	0.049804	-0.023380	0.047924
Correlation C(h), S(h)		0.586093	0.758637	
Regression	b=0.755865	a=0.135233	b=0.953327	a=0.044433

Table 4: Results for g_7 and g_8.

	Results for g_9		Results for g_{10}	
	Mean	Std. Deviation	Mean	Std. Deviation
S(h)	0.575307	0.138837	0.638784	0.137876
C(h)	0.629780	0.123005	0.690128	0.139022
C(h)-S(h)	0.054473	0.047231	0.051344	0.041511
Correlation C(h), S(h)		0.594171	0.679747	
Regression	b=0.670650	a=0.152945	b=0.674144	a=0.173539

Table 5: Results for g_9 and g_{10}.

of the functions (0.50). Generally speaking, it can be said how Choquet integral is more sensitive to modifications of the measure with respect to an uniform reference, and it is also more sensitive to the values of functions h. Sugeno integral performs like an order-based measure; thus, it is more stable than Choquet's one. On the contrary, Choquet integral is similar to the classical arithmetic mean, and so it is more sensitive and variable (note how the differences with respect to the value 0.50 are clearly greater for Choquet integral than for Sugeno integral when the probability measure is "misshaped").

5 Conclusions

Attending the obtained results, it can be said that, according to the way both integrals are calculated, Choquet integral can be viewed as a weighted average, with variability and sensibility of average measures. On the contrary, Sugeno integral seems a generalization of the concept of median. So, the use of Sugeno integral can be suggested in order to try to obtain the measure of "size" of a function, that is, the coincidences between the values of the measure and the values of the function that is being integrated. However, Choquet integral is useful when one is interested on determining the mean values of the functions in the arithmetic sense.

So, it can be deduced from the results that both functionals are complementary. Thus, each integral can lead to the definition of clearly different theoretical systems and practical applications.

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