On Musielak-Orlicz spaces isometric to $L_2$ or $L_\infty$

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ABSTRACT

It is proved that a Musielak-Orlicz space $L_\Phi$ of real valued functions which is isometric to a Hilbert space coincides with $L_2$ up to a weight, that is $\Phi(u,t) = c(t)u^2$. Moreover it is shown that any surjective isometry between $L_\Phi$ and $L_\infty$ is a weighted composition operator and a criterion for $L_\Phi$ to be isometric to $L_\infty$ is presented.

Isometries in complex function spaces have been studied successfully for fairly long time (see review article by Fleming and Jamison [3]). The real case appeared more difficult and since the well known characterization of isometries in $L_p$ spaces by S. Banach in 1932 and some partial results in other spaces, only very recently such isometries in rearrangement invariant real function spaces have been characterized by Kalton and Randrianantoanina in [7]. Isometries between r.i. real spaces and $L_p$ have been also studied in [2]. Here we study surjective linear isometries in the class of Musielak-Orlicz spaces of real-valued functions. We show that any Musielak-Orlicz space $L_\Phi$ isometric to a Hilbert space must coincide with $L_2$ “up to a weight”, namely there exists a positive measurable function $c(t)$, such that $\Phi(u,t) = c(t)u^2$.

We also present a criterion for $L_\Phi$ to be isometric to $L_\infty$, and moreover we show that any such isometry has disjoint support property and in consequence is a weighted composition operator.

In the sequel let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite, complete and nonatomic measure space. Symbols $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{N}$ and $\mathbb{Z}$ stand, as usual, for reals, nonnegative reals, natural numbers and for integers respectively. The space of all (equivalence classes of) $\Sigma$-measurable real functions defined on $\Omega$ is denoted by $L^0$. $L^0$ is a lattice with the pointwise order, that is $f \leq g$ whenever $f(t) \leq g(t)$, a.e. As usual $L_2$ and $L_\infty$ stand
for Lebesgue spaces of 2-integrable functions and for \( \mu \)-essentially bounded functions on \( \Omega \), respectively. A Banach function space \( E \) over the measure space \((\Omega, \Sigma, \mu)\) is a Banach lattice, which as a vector lattice is an ideal of vector lattice \( L^0 \) ([9]).

If \( U : E \to F \) is a linear isometry between Banach function spaces \( E \) and \( F \), then we say that \( U \) has disjoint support property if for any \( f, g \) with \( fg = 0 \) a.e. it holds \( Uf \cdot Ug = 0 \) a.e. A linear operator \( U : E \to F \) is called a weighted composition operator if \( Uf(t) = w(t)f \circ \tau(t) \) a.e. for all \( f \in E \), where \( w : \Omega \to \mathbb{R} \) is \( \Sigma \)-measurable and \( \tau : \Sigma \to \Sigma \) is a set automorphism ([3]). If a weighted composition operator is a surjective isometry, then \( \tau \) is a regular set isomorphism defined modulo null sets on \( \Sigma \) ([4, 5, 6]). Observe that any surjective linear isometry between Banach function spaces with disjoint support property, is a weighted composition operator. Indeed, there exists a partition \( \{\Omega_k\} \) of \( \Omega \) such that \( \chi_{\Omega_k} \in E \) (Cor. 2, p. 95 in [9]). Then setting \( w(t) = U(\chi_{\Omega_k})(t) \) for \( t \in \text{supp} \ U(\chi_{\Omega_k}) \) and \( \tau(A) = \bigcup_{k=1}^{\infty} \text{supp} \ (\chi_{\Omega_k} \cap A) \) we easily show that \( U \) is a weighted composition operator (compare e.g. [6]).

Let \( \Phi(u, t) : \mathbb{R}^+ \times \Omega \to [0, +\infty] \) be a Young function with parameter i.e. for \( t \in \Omega \), \( \Phi(0, t) = 0 \), \( u \mapsto \Phi(u, t) \) is a left continuous convex function, and it is not identically zero or infinity for any \( t \in \Omega \). For a Young function \( \Phi \) with parameter we associate the Musielak-Orlicz space \( L_\Phi \) defined as the subset of \( L^0 \) of all functions \( f \) for which

\[
I_\Phi(\lambda f) = \int_\Omega \Phi(\lambda |f(t)|, t) \, d\mu(t) < \infty
\]

for some \( \lambda > 0 \). \( L_\Phi \) is a Banach space under the Luxemburg norm

\[
||f|| = \inf \{\epsilon > 0 : I_\Phi(f/\epsilon) \leq 1\}.
\]

Define

\[
a(t) = \sup \{u > 0 : \Phi(u, t) < \infty\}.
\]

By the assumptions on \( \Phi \), \( a(t) > 0 \) a.e. on \( \Omega \). More information on Musielak-Orlicz spaces may be found in [12].

Let \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) be convex, \( \varphi(0) = 0 \) and \( \varphi(1) = 1 \). A number \( a > 0 \) is called multiplier of \( \varphi \) if \( \varphi(au) = \varphi(a)\varphi(u) \) for every \( u \in \mathbb{R}^+ \). Let’s denote by \( M_\varphi \) the group of multipliers of \( \varphi \). It was proved by Lamperti in [10] that either \( M_\varphi = \{1\} \) or \( M_\varphi = \{ a^k : k \in \mathbb{Z} \} \) for some \( a > 0 \) or \( M_\varphi = \mathbb{R}^+ \). In the latter case \( \varphi \) is a power function i.e. \( \varphi(u) = u^p \) for some \( 1 \leq p < \infty \).

Further we will need the following measure theoretic lemma from [4].

**Lemma 1**

Let \( \nu \) be a measure on \( \Sigma \) absolutely continuous with respect to the measure \( \mu \). Let \( 0 < c < \nu(\Omega) \) and \( f \) be a measurable real function such that \( \int_A f(t) \, d\nu(t) = 0 \) for any \( A \) with \( \nu(A) = c \). Then \( f = 0 \) a.e. on \( \Omega \).
Theorem 2

If $\Phi(1,t) \equiv 1$ and $L_\Phi$ is isometric to a Hilbert space, then $\Phi(u,t) \equiv u^2$.

Proof. Let’s first observe that $\Phi$ must assume only finite values, since otherwise $L_\Phi$ contains an isomorphic copy of $l_\infty$. Moreover, there exists a disjoint sequence $\{\Omega_i\}$, a partition of $\Omega$, such that

\[ \sup_{t \in \Omega_i} \Phi(u,t) < \infty \]

for every $u \geq 0$ and all $i \in \mathbb{N}$ (see (0.1) in [8]). Then without loss of generality we suppose, that $\int_{\Omega} \Phi(u,t) \, d\mu(t) < \infty$ for every $u > 0$. This, among others, implies that for any function $\Phi$ with $\nu(C) = \frac{1}{2} \int_{\Omega} \Phi(\lambda, t) \, d\mu(t) = 1$ and $\nu(\Omega) = 2$ for any two disjoint sets $A$ and $B$ with $||\lambda \chi_A|| = ||\lambda \chi_B|| = 1$, applying the parallelogram law we get that $||\lambda \chi_A + \lambda \chi_B||^2 = 2$. Hence

\[ 1 = \left\| \frac{\lambda}{\sqrt{2}} \chi_{A \cup B} \right\| = \int_{A \cup B} \Phi\left(\frac{\lambda}{\sqrt{2}}, t\right) \, d\mu(t) = \frac{1}{2} \int_{A \cup B} \Phi(\lambda, t) \, d\mu(t). \]

Thus

\[ \int_{A \cup B} \left[ \Phi\left(\frac{\lambda}{\sqrt{2}}, t\right) - \frac{1}{2} \Phi(\lambda, t) \right] \, d\mu(t) = 0, \]

for any disjoint sets $A$ and $B$ such that $\int_{A} \Phi(\lambda, t) \, d\mu(t) = \int_{B} \Phi(\lambda, t) \, d\mu(t) = 1$. Applying now Lemma 1 for any $C$ with $\nu(C) = \frac{1}{2} \int_{C} \Phi(\lambda, t) \, d\mu(t) = 2 < \nu(\Omega)$, we obtain that $\Phi\left(\frac{\lambda}{\sqrt{2}}, t\right) = \frac{1}{2} \Phi(\lambda, t)$ for a.a. $t \in \Omega$. The same equation holds for any number bigger than $\lambda$. Thus there exists a nonnegative $\beta$ such that for every $\lambda \geq \beta$

\[ \Phi\left(\frac{\lambda}{\sqrt{2}}, t\right) = \frac{1}{2} \Phi(\lambda, t) \]

a.e. on $\Omega$. Now let $\lambda < \beta$. Choose any $0 < c < \min\{1, \frac{1}{2} \int_{\Omega} \Phi(\lambda, t) \, d\mu(t)\}$. For any two disjoint sets $A$ and $B$ with $I_\Phi(\lambda \chi_A) = I_\Phi(\lambda \chi_B) = c$ we choose $\gamma \geq \beta$ and two other sets $A_1$ and $B_1$ such that all sets $A, B, A_1, B_1$ are disjoint and $I_\Phi(\gamma \chi_{A_1}) = I_\Phi(\gamma \chi_{B_1}) = 1 - c$. Setting

\[ f = \lambda \chi_A + \gamma \chi_{A_1} \quad \text{and} \quad g = \lambda \chi_B + \gamma \chi_{B_1}, \]

$||f|| = ||g|| = 1$. Therefore, by the parallelogram law applied to $f$ and $g$ we obtain

\[ \int_{A \cup B} \Phi\left(\frac{\lambda}{\sqrt{2}}, t\right) \, d\mu(t) + \int_{A_1 \cup B_1} \Phi\left(\frac{\gamma}{\sqrt{2}}, t\right) \, d\mu(t) = 1. \]
By the choice of $\gamma$ and the sets $A_1$ and $B_1$, and by the equality $\Phi(\frac{\gamma}{\sqrt{2}}, t) = \frac{1}{2} \Phi(\gamma, t)$,

$$\int_{A_1 \cup B_1} \Phi\left(\frac{\gamma}{\sqrt{2}}, t\right) d\mu(t) = 1 - c.$$ 

Then $\int_{A \cup B} [\Phi(\frac{\lambda}{\sqrt{3}}, t) - \frac{1}{2} \Phi(\lambda, t)] d\mu(t) = 0$ for any disjoint sets $A, B$ with $\int_A \Phi(\lambda, t) d\mu(t) = \int_B \Phi(\lambda, t) d\mu(t) = c$. Applying Lemma 1 again we get that $\Phi(\frac{\lambda}{\sqrt{2}}, t) = \frac{1}{2} \Phi(\lambda, t)$ a.e. on $\Omega$.

The above two steps show that $\frac{1}{\sqrt{2}}$ is a multiplier of the function $u \mapsto \Phi(u, t)$ a.e. on $\Omega$ that is $\Phi(\frac{u}{\sqrt{2}}, t) = \Phi(\frac{1}{\sqrt{2}}, t) \cdot \Phi(u, t)$ for every $u \geq 0$.

Now, let’s sketch only the proof that this function has another multiplier, e.g. $\frac{1}{\sqrt{3}}$. Consider three disjoint sets $A, B$ and $C$ such that $\int_A \Phi(\lambda, t) d\mu(t) = \int_B \Phi(\lambda, t) d\mu(t) = 1$. Then by the parallelogram law applied to the functions $\lambda \chi_A$, $\lambda \chi_B$ and $\lambda \chi_C$, we get that $\|\lambda \chi_{A \cup B \cup C}\| = \sqrt{3}$, whence

$$\int_{A \cup B \cup C} \left[ \Phi\left(\frac{\lambda}{\sqrt{3}}, t\right) - \frac{1}{3} \Phi(\lambda, t) \right] d\mu(t) = 0.$$ 

Further we apply Lemma 1 and proceed analogously as before.

We showed that $u \mapsto \Phi(u, t)$ has two different multipliers and so by the Lamperti’s result ([11]), $\Phi(u, t)$ must be a power function. Finally, since $L_\Phi$ is isometric to a Hilbert space, $\Phi(u, t) \equiv u^2$.  

\textbf{Corollary 3} 

If a Musielak-Orlicz space $L_\Phi$ is isometric to a Hilbert space, then there exists a measurable positive function $c(t)$ such that $\Phi(u, t) = c(t) u^2$.

\textbf{Proof.} Setting $\Phi(u, t) = \Phi(b(t)u, t)$ with $\Phi(b(t), t) = 1$, $L_\Phi$ and $L_{\Phi}$ are isometric. Applying now the previous theorem to $L_{\Phi}$, we get that $\Phi(u, t) \equiv u^2$. Hence $\Phi(u, t) = c(t) u^2$, with $c(t) = \Phi(1, t)$.  

Before we present our last result, let us recall the well known fact, due to Abramovich and Wojtaszczyk, that a Banach lattice $E$ is isomorphic to an $M$-space if and only if there exists $c > 0$ such that $\|x_1 + \ldots + x_n\| \leq c \max\{\|x_1\|, \ldots, \|x_n\|\}$ for all disjoint elements $x_1, \ldots, x_n$ in $E$ (cf. [1], p. 324).

\textbf{Theorem 4} 

A Musielak-Orlicz space $L_\Phi$ is isometric to $L_\infty$ if and only if $I_\Phi(a) \leq 1$. Moreover, if $U : L_\Phi \to L_\infty$ is a surjective isometry, then $U$ has disjoint support property and in consequence $U$ is a weighted composition operator.
Thus and \( Uf = g \) \( \equiv \). Let \( \{ \Omega_i \} \) be a disjoint partition of \( \Omega \) satisfying condition (1). There exist \( \beta > 0 \) and a set \( A \in \Sigma \) such that \( ||\beta a\chi_A|| = 1 \). By the Fatou property of \( L_\Phi \), \( ||\beta a\chi_{A_i}|| \rightarrow ||\beta a\chi_A|| = 1 \). Hence there exists \( n \in \mathbb{N} \) such that \( ||\beta a\chi_{A_i}|| \geq 1/2 \). We have that \( \Phi(\lambda a\chi_{A_i}) < \infty \) for every \( \lambda > 0 \). Therefore, for every \( K > 0 \) there exist \( M \geq K \) and a finite disjoint partition \( \{ A_1, A_2, \ldots, A_m \} \) of \( \bigcup_{i=1}^{n} \Omega_k \) such that \( ||M a\chi_A|| = 1 \) for \( i = 1, 2, \ldots, m \). Thus for an arbitrary large number \( K \) we construct a finite sequence of disjoint functions \( f_i = M a\chi_A, ||f_i|| = 1 \) and \( ||f_1 + f_2 + \ldots + f_m|| \geq K/2 \). Then in view of [1], \( L_\Phi \) cannot be \( M \)-space, and so we obtain a contradiction.

Now let \( I_\Phi(a) > 1 \). Then there exist two sets \( C_1 \) and \( C_2 \) with \( \mu(C_1 \triangle C_2) > 0 \), \( I_\Phi(a\chi_{C_1}) = I_\Phi(a\chi_{C_2}) = 1 \) and \( I_\Phi(a\chi_{C_1} - a\chi_{C_2}) \leq \frac{2}{3} \). The functions \( f = a\chi_{C_1} \) and \( g = a\chi_{C_2} \) are extreme points of the unit ball in \( L_\Phi \) and \( ||f - g|| \leq \frac{3}{2} \). Hence \( Uf \) and \( Ug \) are extreme points of the unit ball in \( L_\infty \). Therefore \( |Uf| \equiv |Ug| \equiv 1 \), Moreover \( Uf \neq Ug \), since \( f \neq g \). Thus there is a set \( C \) with positive measure such that \( Uf(t) \neq Ug(t) \) for all \( t \in C \), whence \( |Uf(t) - Ug(t)| = 2 \) on \( C \). So we obtain a contradiction, since

\[
2 = ||Uf - Ug||_{\infty} = ||f - g|| \leq \frac{3}{2} \]

If \( I_\Phi(a) \leq 1 \), then clearly \( U : f \mapsto \frac{f}{2} \) is an isometry of \( L_\Phi \) onto \( L_\infty \).

Finally we shall show that every onto isometry \( U \) between \( L_\Phi \) and \( L_\infty \) has disjoint support property. Since \( I_\Phi(a) \leq 1 \), any function of the form \( |u(t)|a(t) \), where \( u \) is an unimodular function, is an extreme point of the unit ball in \( L_\Phi \). If \( f \) and \( g \) have disjoint supports and \( f + g \) is an extreme point, then \( f - g \) is also extreme, and so

\[
|Uf + Ug| \equiv 1 \equiv |Uf - Ug|.
\]

Thus \( Uf \) and \( Ug \) have disjoint supports. This implies that if \( u \) is an unimodular function and \( A \) any measurable subset of \([0, 1]\), then there exists \( F \) such that

\[
|U(ua\chi_A)| = \chi_F.
\]

Let now \( A \) and \( B \) be two disjoint sets. Then for any unimodular function \( u \), all functions \( U(ua\chi_A), U(ua\chi_B), U(ua\chi_A + ua\chi_B) \) and \( U(ua\chi_A - ua\chi_B) \) take as values either \( 0 \) or \( 1 \) or \( -1 \). Hence \( U(ua\chi_A) \) and \( U(ua\chi_B) \) have disjoint supports.

For \( f = ua\chi_A \), with an unimodular function \( u \) and \( \mu A > 0 \), and \( g \) with \( |g| \leq a \) and \( \text{supp} \ g \cap A = \emptyset \), we shall show that \( Uf \) and \( Ug \) have disjoint supports. Clearly, \( ||f \mp g|| = 1 \) and \( ||g|| \leq 1 \). Hence \( ||Uf \mp Ug|| = 1 \) and \( ||Ug|| \leq 1 \). Moreover, there
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exists $F$ such that $|Uf| = \chi_F$. Assuming for some $t$ that $Uf(t) = 1$ or $-1$ and $ Ug(t) = b$ where $|b| \leq 1$, we get $|1 \mp b| \leq 1$, whence $b = 0$. Thus $Uf : Ug = 0$.

Now let $f$ and $g$ be such that $|f| \leq a$, $|g| \leq a$, supp $f = A$, supp $g = B$ and $A \cap B = \emptyset$.

Assume first that $A \cup B = \Omega$. By the preceding step, $U(a\chi_A)$ and $U(g)$ have disjoint supports as well as $U(a\chi_B)$ and $U(f)$. Moreover, supp $U(a\chi_A) = C$ and supp $U(a\chi_B) = D$, where $C$ and $D$ are disjoint and their union is the whole set $\Omega$. Hence supp $Ug \subset D$ and supp $Uf \subset C$.

Now let the set $(A \cup B)^c$ have positive measure. Then setting $h = a\chi_{(A \cup B)^c}$ and applying the above paragraph to $f + h$ and $g$, we have that $U(f + h) \cdot Ug = 0$. Since $Ug \cdot Uh = 0$, we have $Uf \cdot Ug = 0$.

Observe, that for any $0 \neq f \in L_q$, $\frac{|f|}{||f||} \leq a$. This completes the proof. □

Let’s mention that for complex Musielak-Orlicz spaces Theorem 4 may be quickly derived from the results in [6].

Acknowledgments. The author would like to thank W. Wnuk for his valuable suggestions and comments.

References